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METHOD OF CHARACTERISTICS FOR THE DYNAMIC THERMOELASTIC PROBLEM OF A CUBICALLY ANISOTROPIC BODY IN STRESSES

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The equations of motion of a two-dimensional cubically anisotropic thermoelastic medium in stress-tensor components have been obtained. The equation of characteristics has been derived and the dependences of the phase and group velocities of propagation of discontinuity surfaces on the slope of the normal to the characteristic surface and the period of relaxation of the heat flux have been investigated using this equation.

Introduction. The dynamic problems of the theory of thermoelasticity of isotropic and anisotropic bodies play an important role in investigating the processes of deformation of continuous media. However in most cases their solution is associated with the use of the theory of plane waves and is made difficult by the fact that dispersion equations have a cumbersome form and can be solved analytically just for special directions or planes. Thus, the propagation of one-dimensional waves in an isotropic medium has been investigated in [1], while the propagation of one-dimensional and two-dimensional waves in elastic anisotropic bodies with allowance for the relaxation time of the heat flux has been considered in [2, 3]. Of undoubted interest are [4, 5], where results of investigations of wave processes in a thermoelastic medium using the method of characteristics are given. Below we consider implementation of the method of characteristics as applied to an analysis of the regularities of propagation of discontinuity surfaces in a two-dimensional cubically anisotropic medium in the plane $x_3 = 0$.

Method of Characteristics. To describe the dynamic processes in the plane $x_3 = 0$ of a cubically anisotropic medium we write the Duhamel–Neumann law in the following form [2, 3]:

$$\sigma_{11} = A_1 e_{11} + A_2 e_{22} - \beta T\,,\ \ \sigma_{22} = A_2 e_{11} + A_1 e_{22} - \beta T\,,\ \ \sigma_{33} = A_2\,(e_{11} + e_{22}) - \beta T\,,\ \ \sigma_{12} = \sigma_{21} = 2A_4 e_{12}\,, \tag{1}$$

where $e_{kl} = \frac{1}{2}(\partial_l u_k + \partial_k u_l)$ is the deformation tensor; $\partial_k = \partial/\partial x_k$, k, $l = 1, 2$.

Expressions (1) yield the following equations of motion [2, 3]:

$$(A_4\Delta + \varepsilon\partial_i^2)\,u_i + (A_2 + A_4)\,\partial_i \sum_{k=1}^{2} \partial_k u_k + X_i = \rho\ddot{u}_i + \beta\partial_i T\,,\quad i = 1, 2\,, \tag{2}$$

here ρ is the density, X_i are the mass forces, $\ddot{u}_i = \partial^2 u_i/\partial t^2$, $i = 1, 2$, and $\varepsilon = A_1 - A_2 - 2A_4$.

To pass to the equations of motion in stresses we take into account that the independent components in (1) are σ_{11}, σ_{12}, and σ_{22}. We differentiate (2) with respect to x_i and x_j and add them together. As a result we obtain

$$A_4\Delta\,(\partial_i u_j + \partial_j u_i) + \varepsilon\,(\partial_j\partial_i^2 u_i + \partial_i\partial_j^2 u_j) + 2\,(A_2 + A_4)\,\partial_j\partial_i \sum_{k=1}^{2} \partial_k u_k + \partial_j X_i + \partial_i X_j = \rho\,(\partial_j\ddot{u}_i + \partial_i\ddot{u}_j) + 2\beta\partial_i\partial_j T\,,$$

or

$$2A_4\Delta e_{ij} + \varepsilon\partial_i\partial_j\,(e_{ii} + e_{jj}) + 2\,(A_2 + A_4)\,\partial_j\partial_i \sum_{k=1}^{2} e_{kk} + \partial_j X_i + \partial_i X_j = 2\rho\ddot{e}_{ij} + 2\beta\partial_i\partial_j T\,. \tag{3}$$

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Expressions (1) yield the following relations for the deformation tensor:

$$e_{11} = \frac{A_1\sigma_{11} - A_2\sigma_{22}}{A_1^2 - A_2^2} + \frac{\beta T}{A_1 + A_2}, \quad e_{22} = \frac{A_1\sigma_{22} - A_2\sigma_{11}}{A_1^2 - A_2^2} + \frac{\beta T}{A_1 + A_2},$$

$$e_{11} + e_{22} = \frac{\sigma_{11} + \sigma_{22}}{A_1 + A_2} + \frac{2\beta T}{A_1 + A_2}, \quad e_{12} = \frac{\sigma_{12}}{2A_4}. \tag{4}$$

We introduce (4) into system (3) for the cases where $i = j = \overline{1, 2}$ and $i = 1$, $j = 2$. Upon obvious transformations we will have

$$A_1 (A_4\Delta\sigma_{11} - \rho\ddot{\sigma}_{11}) + (A_1 + A_2)(A_1 - A_2 - 2A_4)\,\partial_1^2\sigma_{11} - A_2 (A_4\Delta\sigma_{22} - \rho\ddot{\sigma}_{22}) + A_4 (A_1 + A_2)\,\partial_1^2\sigma_{22} + \partial_1 X_1 +$$
$$+ \beta (A_1 - A_2)(A_4\Delta T - \rho\ddot{T}) = 0,$$

$$A_4\Delta\sigma_{12} - \rho\ddot{\sigma}_{12} + A_4\partial_1\partial_2 (\sigma_{11} + \sigma_{22}) + \partial_2 X_1 + \partial_1 X_2 = 0, \tag{5}$$

$$A_1 (A_4\Delta\sigma_{22} - \rho\ddot{\sigma}_{22}) + (A_1 + A_2)(A_1 - A_2 - 2A_4)\,\partial_2^2\sigma_{22} - A_2 (A_4\Delta\sigma_{11} - \rho\ddot{\sigma}_{11}) + A_4 (A_1 + A_2)\,\partial_2^2\sigma_{11} + \partial_2 X_2 +$$
$$+ \beta (A_1 - A_2)(A_4\Delta T - \rho\ddot{T}) = 0.$$

To obtain a complete system of equations of generalized interconnected thermoelasticity for a cubically anisotropic body in the plane $x_3 = 0$ we use the following heat-conduction equation [3, 4]:

$$\lambda\Delta T - c_v (\dot{T} + \tau\ddot{T}) = T_0 \beta \left(\sum_{k=1}^{2} \dot{e}_{kk} + \tau \sum_{k=1}^{2} \ddot{e}_{kk} \right), \tag{6}$$

From (6), with account for (4), we obtain

$$\lambda\Delta T - (\dot{T} + \tau\ddot{T}) \left(c_v + \frac{2\beta^2 T_0}{A_1 + A_2} \right) - \frac{\beta T_0}{A_1 + A_2} (\dot{\sigma}_{11} + \dot{\sigma}_{22} + \tau (\ddot{\sigma}_{11} + \ddot{\sigma}_{22})) = 0. \tag{7}$$

Here $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}$.

We prescribe the initial data for system (5) and (7) on the surface $Z = Z_0(t, x_1, x_2)$ and pass to new variables according to the following scheme [5]:

$$Z = Z_0 (t, x_1, x_2), \quad Z_k = Z_k (t, x_1, x_2), \quad k = 1, 2.$$

Then

$$\frac{\partial y (t, X)}{\partial x_k} = \sum_{i=0}^{2} \frac{\partial y}{\partial Z_i} \frac{\partial Z_i}{\partial x_k}, \quad \frac{\partial^2 y}{\partial x_k \partial x_n} = \sum_{i,j=0}^{2} \frac{\partial^2 y}{\partial Z_j \partial Z_i} \frac{\partial Z_i}{\partial x_k} \frac{\partial Z_j}{\partial x_n} + \sum_{i=0}^{2} \frac{\partial y}{\partial Z_i} \frac{\partial^2 Z_i}{\partial x_n \partial x_k}. \tag{8}$$

We introduce (8) into Eqs. (5) and (7) and write the terms containing the derivatives $\partial^2\sigma_{ij}/\partial Z^2$ and $\partial^2 T/\partial Z^2$, $i, j = 1, 2$ [5]. We will have

$$(A_1(A_4g^2-\rho p_0^2)+(A_1+A_2)(A_1-A_2-2A_4)p_1^2)\frac{\partial^2\sigma_{11}}{\partial Z^2}-(A_2(A_4g^2-\rho p_0^2)-A_4(A_1+A_2)p_1^2)\frac{\partial^2\sigma_{22}}{\partial Z^2}+$$

$$+\beta(A_1-A_2)(A_4g^2-\rho p_0^2)\frac{\partial^2 T}{\partial Z^2}+\ldots=0,$$

$$(A_4g^2-\rho p_0^2)\frac{\partial^2\sigma_{12}}{\partial Z^2}+A_4p_1p_2\left(\frac{\partial^2\sigma_{11}}{\partial Z^2}+\frac{\partial^2\sigma_{22}}{\partial Z^2}\right)+\ldots=0,$$

$$(A_1(A_4g^2-\rho p_0^2)+(A_1+A_2)(A_1-A_2-2A_4)p_1^2)\frac{\partial^2\sigma_{22}}{\partial Z^2}-(A_2(A_4g^2-\rho p_0^2)-A_4(A_1+A_2)p_1^2)\frac{\partial^2\sigma_{11}}{\partial Z^2}+$$

$$+\beta(A_1-A_2)(A_4g^2-\rho p_0^2)\frac{\partial^2 T}{\partial Z^2}+\ldots=0,$$

$$\left(\lambda g^2-\tau p_0^2\left(c_v+\frac{2\beta^2T_0}{A_1+A_2}\right)\right)\frac{\partial^2 T}{\partial Z^2}-\frac{\beta T_0\tau p_0^2}{A_1+A_2}\left(\frac{\partial^2\sigma_{11}}{\partial Z^2}+\frac{\partial^2\sigma_{22}}{\partial Z^2}\right)+\ldots=0,$$

where $g^2=p_1^2+p_2^2$; $p_0=\partial Z/\partial t$, and $p_k=\partial Z/\partial x_k$, $k=1,2$. The equation of the characteristic surface will be obtained from the condition of insolubility of the last system of equations relative to the derivatives of second order in Z [5]. This is equivalent to the equality to zero of the determinant which is composed of the coefficients of these derivatives. By expanding this determinant we obtain

$$(A_4g^2-\rho p_0^2)(\beta b\tau p_0^2(A_4g^2-\rho p_0^2)((A_1-A_2)g^2-2\rho p_0^2)+$$

$$+(\lambda g^2-(c_v+a)p_0^2)((A_4g^2-\rho p_0^2)(A_1g^2-\rho p_0^2)+\varepsilon(A_1+A_2)p_1^2p_2^2))=0. \tag{9}$$

Hence for the phase velocities of propagation of elastic and thermoelastic waves we will have

$$A_4=\rho V^2, \tag{10}$$

$$\beta b\tau V^2(A_4-\rho V^2)((A_1-A_2)-2\rho V^2)+(\lambda-(c_v+a)V^2)((A_4-\rho V^2)(A_1-\rho V^2)+$$

$$+\varepsilon(A_1+A_2)\cos^2\alpha\sin^2\alpha)=0, \tag{11}$$

where $a=2\beta^2T_0(A_1+A_2)$; $b=\beta T_0(A_1+A_2)$; $V^2=p_0^2/g^2$; $\cos^2\alpha=p_1^2/g^2$ is the square of the direction cosine of the normal to the characteristic surface [5]. From (10) we will have

$$V^*=\sqrt{\frac{A_4}{\rho}}.$$

We rewrite Eq. (11) as follows:

$$V^6+B_1V^4+(B_2+B_3\cos^2\alpha\sin^2\alpha)V^2-(B_4+B_5\cos^2\alpha\sin^2\alpha)=0, \tag{12}$$

where

TABLE 1. Connectivity Constants a and b in the Plane $x_3 = 0$ of Some Cubically Anisotropic Media

Material	Elastic constants $\times 10^{10}$, N/m^2			$\alpha_t \cdot 10^{-6}$, 1/deg	a, kN/(deg·m^2)	b
	A_1	A_2	A_4			
Silver	12.4	9.34	4.61	19.0	98.623	0.0056
Lead	4.66	3.92	1.44	28.35	88.309	0.0083
Molybdenum	46	17.6	11.0	5.0	17.844	0.0015
Aluminum	10.82	6.13	2.85	22.6	94.063	0.00902
Gold	18.6	15.7	4.20	14.0	83.714	0.006
Copper	16.84	12.14	7.54	16.61	94.329	0.007
Nickel	24.65	14.73	12.47	12.55	68.622	0.0051
Tungsten	50.1	19.8	15.14	4.4	13059	0.0017

$$B_1 = \frac{\tau\left((A_1 - A_2 + 2A_4)\, b\beta - (A_1 + A_4)\,(c_v + a)\right) - \lambda\rho}{\rho\tau c_v}\,;$$

$$B_2 = \frac{\lambda\rho\,(A_1 + A_4) + \tau\,(c_v + a)\,A_1 A_4 - \beta b\tau A_4\,(A_1 - A_2)}{\rho^2 \tau c_v}\,; \quad B_3 = \frac{\varepsilon\,(c_v + a)\,(A_1 + A_2)}{\rho^2 c_v}\,;$$

$$B_4 = \frac{\lambda A_1 A_4}{\rho^2 \tau c_v}\,; \quad B_5 = \frac{\lambda\varepsilon\,(A_1 + A_2)}{\rho^2 \tau c_v}\,.$$

For the expressions for the velocities of propagation of thermoelastic waves to be obtained from (12) we introduce the following replacement:

$$p = B_2 - \frac{B_1^2}{3} + B_3 \cos^2\alpha \sin^2\alpha\,, \quad q = \frac{2B_1^3}{27} - \frac{B_1}{3}\,(B_2 + B_3 \cos^2\alpha \sin^2\alpha) - B_4 - B_5 \cos^2\alpha \sin^2\alpha\,. \tag{13}$$

Equation (12) with account for (13) will be written in the form

$$V^6 + pV^2 + q = 0\,.$$

Hence on the basis of the existing formulas for the roots of a reduced cubic equation [6] we obtain

$$V_k = \left(-\frac{B_1}{3} + 2\sqrt{-\frac{p}{3}}\cos\left[\frac{1}{3}\,(\varphi + 2\pi k)\right]\right)^{1/2}, \quad \varphi = \arccos\left(-\frac{q}{2}\sqrt{-\left(\frac{3}{p}\right)^3}\right), \quad k = \overline{1, 3}\,. \tag{14}$$

It is expedient to investigate the dependences of the velocities of propagation of thermoelastic waves V_k, $k = \overline{1, 3}$, on the slope (angle of inclination) α of the normal to the characteristic surface as compared to the velocities of propagation of elastic waves V_1^{el} and V_2^{el} and a thermoelastic wave V_{t} which are determined from (11) in the absence of the effect of connectivity of the mechanical and thermal fields ($\tau b \approx 0$) as follows:

$$V_{1,2}^{\mathrm{el}} = \left(\frac{A_1 + A_4 \pm \left((A_1 - A_4)^2 - (A_1 - A_2 - 2A_4)\,(A_1 + A_2)\sin^2 2\alpha\right)^{1/2}}{2\rho}\right)^{1/2}, \tag{15}$$

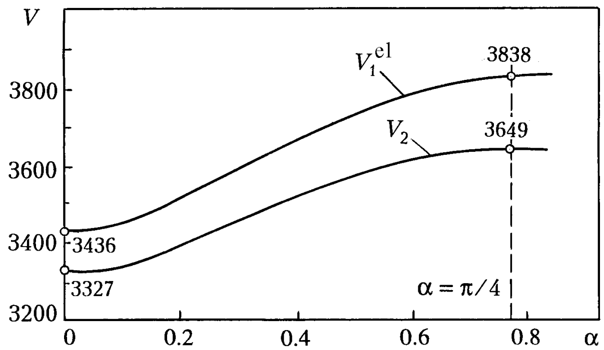
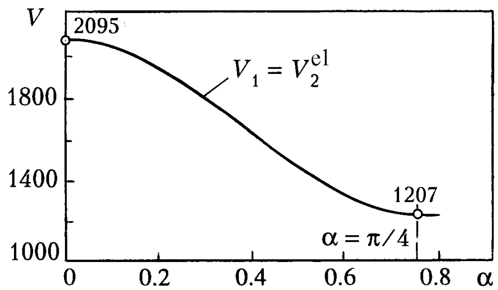


Fig. 1. Dependences $V_2(\alpha)$ and $V_1^{el}(\alpha)$. V, m/sec; α, rad.



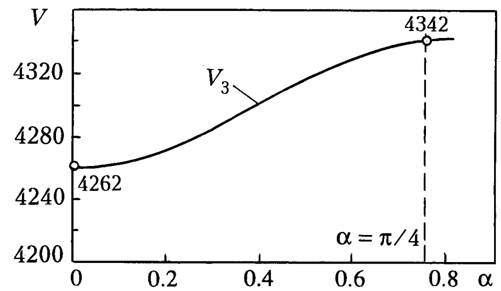


Fig. 2. Dependences $V_1(\alpha)$ and $V_2^{el}(\alpha)$. V, m/sec; α, rad.

Fig. 3. Dependence $V_3(\alpha)$. V, m/sec; α, rad.

$$V_t = \sqrt{\frac{\lambda}{\tau\,(c_v + a)}}\ . \tag{16}$$

We note that the assumption of a negligibly small interconnection of the mechanical field and the temperature field is justified, since the calculation carried out for a number of cubically anisotropic materials at $T_0 = 293$ K (lead, silver, molybdenum, etc.) shows that the constant b has the order of 10^{-3} (Table 1, [7, 8]). The relaxation time of the heat flux for metals is taken to be $1 \cdot 10^{-11}$ sec [4].

Propagation of Discontinuity Surfaces. Let us investigate the dependences of the velocities V_2 and V_1^{el} (Fig. 1), V_1 and V_2^{el} (Fig. 2), and V_3 (Fig. 3) on the slope α of the normal to the characteristic surface for silver ($c_v = 2454$ kJ/(deg·m^3), $\rho = 10505$ kg/m^3, and $\lambda = 418$ W/(deg·m) [7, 8]).

As follows from Fig. 1, the thermoelastic wave propagating with velocity V_2 in the plane $x_3 = 0$ is an elastic wave which is accompanied by a thermal field; the temperature field causes the velocity V_2 to significantly decrease as compared to the velocity V_1^{el} of an elastic quasilongitudinal wave. Therefore, the presence of the temperature field in silver causes the longitudinal deformations to appear. We note that the velocities differ to the greatest extent for $\alpha = \pi/4$ when the difference between V_1^{el} and V_2 is 189 m/sec, which corresponds to a relative decrease of 5% in the velocity V_2 as compared to V_1^{el}.

The comparison of the velocities V_1 and V_2^{el} shows that the temperature field exerts no appreciable influence on the quantity V_1; therefore, this type of discontinuity surface can be considered to be an elastic quasitransverse wave. Taking this into account, we can consider that for silver a temperature change does not lead to the appearance of shear deformations in the plane $x_3 = 0$ of a cubically anisotropic body. From Fig. 3 it follows that the velocity of propagation of the thermoelastic wave V_3 explicitly depends on the slope of the normal to the characteristic surface and increases appreciably as α changes from 0 to $\pi/4$. Comparison of it to the velocity V_t whose value for silver is 4047 m/sec shows that the interconnection of the mechanical and thermal fields substantially influences the propagation of this type of discontinuity surface.

Let us find the group velocities of propagation of the elastic and thermoelastic waves [5]:

$$P = \sqrt{\left(\frac{\partial p_0}{\partial p_1}\right)^2 + \left(\frac{\partial p_0}{\partial p_2}\right)^2} \,. \tag{17}$$

We express p_0 from Eq. (9) as

$$p_0^* = g\sqrt{\frac{A_4}{\rho}} \,, \tag{18}$$

$$p_0^{(k)} = \left(-\frac{gB_1}{3} + 2\left(-\frac{p^*}{3}\right)^{1/2} \cos\left[\frac{1}{3}(\varphi^* + 2\pi k)\right]\right)^{1/2}, \quad \varphi^* = \arccos\left(-\frac{q^*}{2}\left(-\left(\frac{3}{p^*}\right)^3\right)^{1/2}\right), \tag{19}$$

where

$$p^* = \left(B_2 - \frac{B_1^2}{3}\right)g^4 + B_3 p_1^2 p_2^2 \,; \; q^* = \left(\frac{2B_1^3}{27} - B_4\right)g^6 - \frac{B_1 g^2}{3}(B_2\, g^4 + B_3\, p_1^2 p_2^2) - B_5\, p_1^2 p_2^2 g^2 \,, \quad k = \overline{1, 3} \,.$$

We differentiate (18) and (19) with respect to p_1 and p_2. Taking into account that $p_1 = g\cos\alpha$ and $p_2 = g\sin\alpha$, we will have

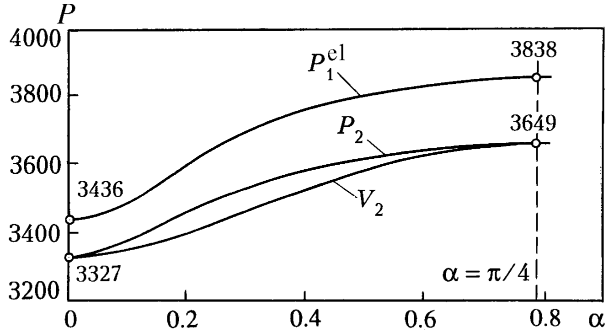
$$\frac{\partial p_0^*}{\partial p_1} = \sqrt{\frac{A_4}{\rho}}\cos\alpha \,, \quad \frac{\partial p_0^*}{\partial p_2} = \sqrt{\frac{A_4}{\rho}}\sin\alpha \,, \tag{20}$$

$$\frac{\partial p_0^{(k)}}{\partial p_1} = \frac{1}{2V_k}\left(\frac{1}{\sqrt{-3p}}\left(p\sin\left[\frac{1}{3}(\varphi + 2\pi k)\right]\frac{r_1}{3\left(1 + \frac{q^2}{4}\frac{27}{p^3}\right)^{1/2}} - 2\cos\alpha\left(B_2 - \frac{2}{3}B_1^2 + B_3\sin^2\alpha\right)\times\right.\right.$$
$$\left.\left.\times\cos\left[\frac{1}{3}(\varphi + 2\pi k)\right]\right) - \frac{2}{3}B_1\cos\alpha\right), \tag{21}$$

$$\frac{\partial p_0^{(k)}}{\partial p_2} = \frac{1}{2V_k}\left(\frac{1}{\sqrt{-3p}}\left(p\sin\left[\frac{1}{3}(\varphi + 2\pi k)\right]\frac{r_2}{3\left(1 + \frac{q^2}{4}\frac{27}{p^3}\right)^{1/2}} - 2\sin\alpha\left(B_2 - \frac{2}{3}B_1^2 + B_3\cos^2\alpha\right)\times\right.\right.$$
$$\left.\left.\times\cos\left[\frac{1}{3}(\varphi + 2\pi k)\right]\right) - \frac{2}{3}B_1\sin\alpha\right), \tag{22}$$

where

$$r_1 = 2\cos\alpha\left(-\left(\frac{3}{p}\right)^3\right)^{1/2}\left(\frac{2}{9}B_1^3 - 3B_4 - \frac{2}{3}B_1(B_2 + B_3(1 + \cos^2\alpha)\sin^2\alpha) -\right.$$



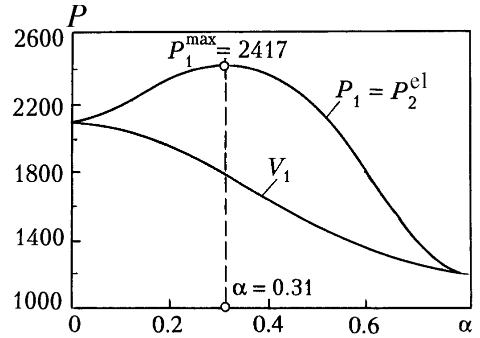


Fig. 4. Dependences $V_2(\alpha)$, $P_2(\alpha)$, and $P_1^{el}(\alpha)$. V, m/sec; P, m/sec; α, rad.

Fig. 5. Dependences $P_1(\alpha)$, $V_1(\alpha)$, and $P_2^{el}(\alpha)$. V, m/sec; P, m/sec; α, rad.

$$-\frac{1}{3}B_1(2B_2+B_3\sin^2\alpha)-B_4\sin^2\alpha(1+\cos^2\alpha)-\frac{3q}{2p}\left(2B_2-\frac{2}{3}B_1^2+B_5\sin^2\alpha\right)\Bigg);$$

$$r_2=2\sin\alpha\left(-\left(\frac{3}{p}\right)^3\right)^{1/2}\Bigg(\frac{2}{9}B_1^3-3B_4-\frac{2}{3}B_1(B_2+B_3(1+\sin^2\alpha)\cos^2\alpha)-$$

$$-\frac{1}{3}B_1(2B_2+B_3\cos^2\alpha)-B_4\cos^2\alpha(1+\sin^2\alpha)-\frac{3q}{2p}\left(2B_2-\frac{2}{3}B_1^2+B_5\cos^2\alpha\right)\Bigg).$$

With account for (17), from (20) we will have $P^*=\sqrt{A_4/\rho}$. In the remaining cases the group velocities of the discontinuity surfaces will be written in the form

$$P_k=\left(\left(\frac{\partial p_0^{(k)}}{\partial p_1}\right)^2+\left(\frac{\partial p_0^{(k)}}{\partial p_2}\right)^2\right)^{1/2}. \tag{23}$$

We compare the dependences of the group velocities of propagation of thermoelastic waves P_k, $k=\overline{1,3}$, on the slope α of the normal to the characteristic surface with the corresponding dependences of the velocities of propagation of elastic waves P_1^{el} and P_2^{el} which are determined according to (23) by the following expressions:

$$\left(\frac{\partial p_0^{(1,2)}}{\partial p_1}\right)^{el}=\sqrt{\frac{1}{2\rho}}\frac{A_1+A_4\pm\dfrac{(A_1-A_4)^2-2\varepsilon(A_1+A_2)\sin^2\alpha}{\sqrt{(A_1-A_4)^2-\varepsilon(A_1+A_2)\sin^2 2\alpha}}}{\sqrt{A_1+A_4\pm\sqrt{(A_1-A_4)^2-\varepsilon(A_1+A_2)\sin^2 2\alpha}}}\cos\alpha\,, \tag{24}$$

$$\left(\frac{\partial p_0^{(1,2)}}{\partial p_2}\right)^{el}=\sqrt{\frac{1}{2\rho}}\frac{A_1+A_4\pm\dfrac{(A_1-A_4)^2-2\varepsilon(A_1+A_2)\cos^2\alpha}{\sqrt{(A_1-A_4)^2-\varepsilon(A_1+A_2)\sin^2 2\alpha}}}{\sqrt{A_1+A_4\pm\sqrt{(A_1-A_4)^2-\varepsilon(A_1+A_2)\sin^2 2\alpha}}}\sin\alpha\,. \tag{25}$$

It is easy to see that the group velocity P_t is determined by formula (16). We plot the group and phase velocities versus the slope of the normal to the characteristic surface with the example of silver (Figs. 4–6).

As follows from Fig. 4, the group velocity of propagation of a thermoelastic wave P_2 is equal to the phase velocity V_2 for $\alpha=\pi n/4$, $n\in Z$; for other values of the slope α the group velocity is higher than the phase velocity. However the velocity P_2 is much lower than P_1^{el}, i.e., the presence of the temperature field leads to a decrease in both the phase velocity and the group velocity of propagation of a discontinuity surface of this type (Figs. 1 and 4).

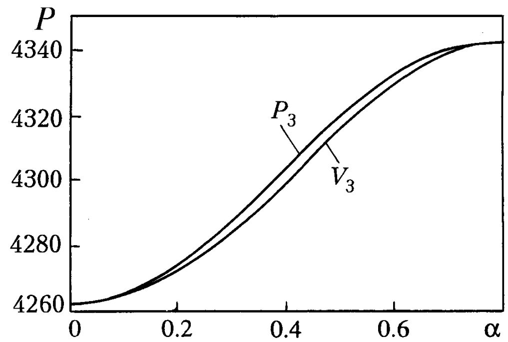


Fig. 6. Dependences $P_3(\alpha)$ and $V_3(\alpha)$. V, m/sec; P, m/sec; α, rad.

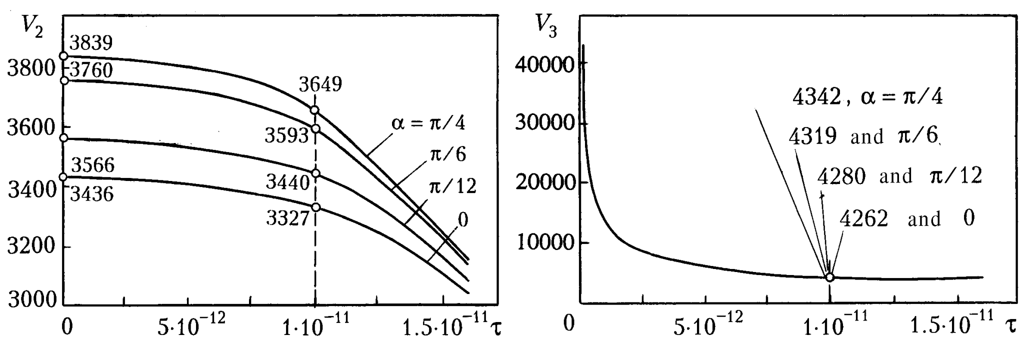


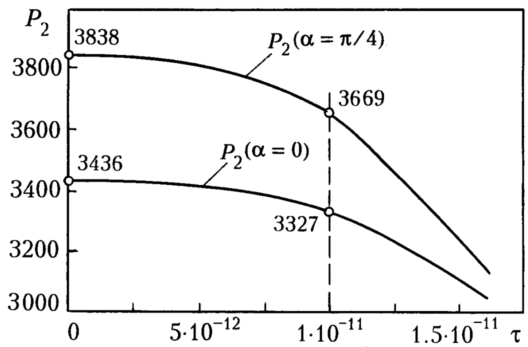
Fig. 7. Dependences $V_2(\tau)$ and $V_3(\tau)$ in the plane $x_3 = 0$. V, m/sec; τ, sec.

The velocity P_1 coincides with P_2^{el}, i.e., this discontinuity surface is a purely elastic quasitransverse wave and in the plane $x_3 = 0$ of a cubically anisotropic body its propagation is not associated with a temperature change. It should be noted that P_1 is much higher than the phase velocity of propagation of the discontinuity surface and for a certain value of the slope of the normal to the characteristic surface the group velocity P_1 is maximum (Fig. 5). The dependences $P_3(\alpha)$ and $V_3(\alpha)$ are analogous and the inequality $P_3(\alpha) \geq V_3(\alpha)$ is fulfilled between them for any values of the slope α (Fig. 6).

Influence of the Relaxation Time of the Heat Flux on the Phase and Group Velocities of Propagation of Discontinuity Surfaces. It should be noted that no exact period of relaxation of the heat flux has been established for metals [9] and one sometimes takes $\tau = 0.5 \cdot 10^{-11}$ sec along with a value of τ equal to $1 \cdot 10^{-11}$ sec in the calculations [10]. Therefore, it is of interest to investigate the influence of the period of relaxation of the heat flux on the phase velocity of thermoelastic waves. Let us consider the plots of V_2 and V_3 versus τ (Fig. 7).

As follows from Fig. 7, when $\tau \to 0$ the velocity of the thermoelastic wave V_2 tends to a finite limit; the curves $V_2(\tau)$ have different limits depending on the slope of the normal to the characteristic surface, and the value of the limiting velocity increases with α. This confirms the idea that the type of the crystal lattice of an anisotropic material (of a cubically anisotropic material in this case) is determining for the value of the period of relaxation of the heat flux [9]. The dependence of the thermoelastic-wave velocity V_3 on τ is unaffected, in practice, by the slope of the normal to the characteristic surface. Whereas for τ of the order of 10^{-11} sec the difference in its values is about 20–30 m/sec, for $\tau \sim 10^{-12}$ sec they do not differ for different slopes α; when $\tau \to 0$ the velocity V_3 becomes infinitely high (Fig. 7). It should be added that a change in the period of relaxation of the heat flux exerts no influence on the values of the velocity V_1.

We note that when the periods of relaxation of the heat flux are extremely short the values of the velocities $V_2(0)$ and $V_2(\pi/4)$ (Fig. 4) are very much like the velocities of propagation $V_1^{el}(0)$ and $V_1^{el}(\pi/4)$ respectively (Fig. 1). This circumstance and the fact that the dependences $V_3(\tau)$ and $V_t(\tau)$ virtually coincide when $\tau \to 0$ enable us to interpret the limiting transition to a zero time of relaxation of the heat flux as the transition from the generalized interconnected dynamic problem of thermoelasticity of a cubically anisotropic body to an unconnected problem where we have two elastic waves and one thermoelastic wave.



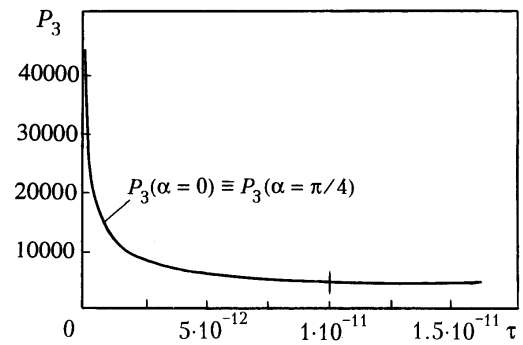


Fig. 8. Dependence $P_2(\tau)$. P, m/sec; τ, sec.

Fig. 9. Dependence $P_3(\tau)$. P, m/sec; τ, sec.

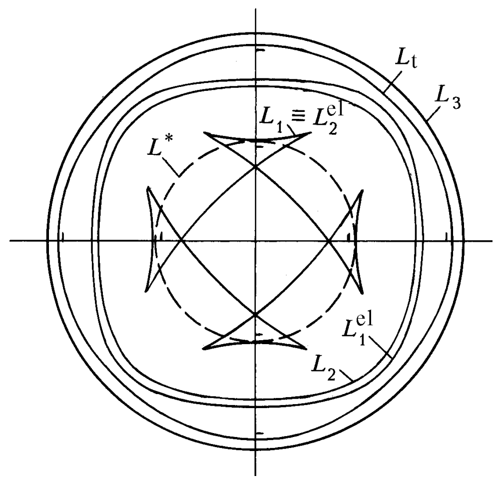


Fig. 10. Wave fronts in the plane $x_3 = 0$. The scale is 1:2000 m.

Let us consider the influence of the relaxation time of the heat flux on the change in the group velocities of propagation of discontinuity surfaces.

From Figs. 8 and 9 it follows that in the plane $x_3 = 0$ of a cubically anisotropic body the dependences of the group velocities $P_2(\tau)$ and $P_3(\tau)$ are analogous to the dependences of the phase velocities V_2 and V_3 on τ. Thus, when $\tau \to 0$ the velocity P_2 tends to a finite limit very much like the value of P_1^{el} for the corresponding slope of the normal to the characteristic surface (Figs. 4 and 8). Unlike P_2, the dependence $P_3(\tau)$ is unaffected, in practice, by the selected direction of propagation of the discontinuity surface, and the velocity P_3 increases without bound with decrease in τ regardless of the slope α (Fig. 9).

Wave Fronts of Discontinuity Surfaces. The regularities of propagation of discontinuity surfaces in the plane $x_3 = 0$ of a cubically anisotropic body clearly characterize their wave fronts, which can be constructed using expressions (20)–(22). We have

$$\frac{dx_j}{dt} = \frac{\partial p_0^*}{\partial p_j}, \quad \frac{dx_j^{(k)}}{dt} = \frac{\partial p_0^{(k)}}{\partial p_j}, \quad j = 1, 2; \quad k = \overline{1, 3}.$$

Hence for $t = 1$ we obtain

$$x_1^2 + x_2^2 = \frac{A_4}{\rho}, \tag{26}$$

$$x_j^{(k)} = \frac{\partial p_0^{(k)}}{\partial p_j}, \quad j = 1, 2; \quad k = \overline{1, 3}. \tag{27}$$

Thus, in our case the characteristic surface at any instant of time consists of four curves, one of which is the circle L^* of radius $\sqrt{A_4/\rho}$ while the other three are prescribed parametrically in the form $(x_1^{(k)}, x_2^{(k)})$, $k = \overline{1, 3}$. The wave fronts of the discontinuity surfaces in the plane $x_3 = 0$ of a cubically anisotropic body are given in Fig. 10.

As follows from Fig. 10, in propagation of thermoelastic waves, the front L_3 leads L_t, which is a circle of radius 4027 m, and conversely, the front L_2 lags behind L_1^{el}. The elastic wave L_1 is propagating with the occurrence of lacuna loops formed by intersection of the branches of this curve.

Conclusions. We note that silver belongs to cubically anisotropic materials in which $A_1 - A_2 - 2A_4 < 0$ (lead, brass, nickel, copper, etc.), which substantially influences the regularities of propagation of waves in continuous media characterized by three elastic constants. In the case where $A_1 - A_2 - 2A_4 < 0$ (molybdenum, tungsten, etc.) the dependences of the phase and group velocities on the slope of the normal to the characteristic surface and the wave fronts differ from those presented in Figs. 1–10.

NOTATION

$\mathbf{u} = (u_1, u_2, u_3)$, displacement vector; A_1, A_2, and A_4, elastic constants; β, thermoelastic constant; $\beta = \alpha_t(A_1 + 2A_2)$; α, coefficient of linear thermal expansion; T, absolute temperature; λ, thermal conductivity; τ, relaxation time of the heat flux; c_v, specific heat at constant volume; T_0, initial temperature.

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